

UNIVERSAL DEFORMATION RINGS OF MODULES FOR ALGEBRAS OF DIHEDRAL TYPE OF POLYNOMIAL GROWTH

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ABSTRACT. Let k be an algebraically closed field, and let Λ be an algebra of dihedral type of polynomial growth as classified by Erdmann and Skowroński. We describe all finitely generated Λ -modules V whose stable endomorphism rings are isomorphic to k and determine their universal deformation rings $R(\Lambda, V)$. We prove that only three isomorphism types occur for $R(\Lambda, V)$: k , $k[[t]]/(t^2)$ and $k[[t]]$.

1. INTRODUCTION

Let k be an algebraically closed field of arbitrary characteristic, and let Λ be a finite dimensional algebra over k . Given a finitely generated Λ -module V , it is a natural question to ask over which complete local commutative Noetherian k -algebras R with residue field k the module V can be lifted. If Λ is self-injective and the stable endomorphism ring of V is isomorphic to k , then there exists a particular complete local commutative Noetherian k -algebra $R(\Lambda, V)$ with residue field k and a particular lift $U(\Lambda, V)$ of V over $R(\Lambda, V)$ which is universal with respect to all isomorphism classes of lifts (i.e. deformations) of V over all such k -algebras R (see [6] and §2). The ring $R(\Lambda, V)$ is called the universal deformation ring of V and $U(\Lambda, V)$ is called the universal deformation of V . Since for every R and every lift M of V over R there exists a unique specialization morphism $R(\Lambda, V) \rightarrow R$ with respect to which $U(\Lambda, V)$ specializes to M , the pair $(R(\Lambda, V), U(\Lambda, V))$ encompasses all lifts of V over all complete local commutative Noetherian k -algebras with residue field k .

The question of lifting modules has a long tradition when Λ is equal to the group ring kG of a finite group G and k has positive characteristic p . In this case, one not only studies lifts of V to complete local commutative Noetherian k -algebras but to arbitrary complete local commutative Noetherian rings with residue field k , thus obtaining more information about the connections between characteristic p and characteristic 0 representations of G . Using Morita equivalence classes of algebras, the case of kG is tightly linked to understanding lifts of modules over arbitrary finite dimensional k -algebras Λ . In particular, if kG (or a block of kG) is Morita equivalent to Λ and V corresponds to V_Λ under this Morita equivalence, then V has a universal mod p deformation ring $R(G, V)/pR(G, V)$ if and only if V_Λ has a universal deformation ring $R(\Lambda, V_\Lambda)$. Moreover, $R(\Lambda, V_\Lambda) \cong R(G, V)/pR(G, V)$. The main advantage of employing this connection is that one can use powerful tools from the representation theory of finite dimensional algebras, such as Auslander-Reiten quivers, stable equivalences, and combinatorial descriptions of modules, just to name a few. This approach has recently led to the solution of various open problems. For example, it was successfully used in [3, 4, 5] to construct group representations whose universal deformation rings are not complete intersections, thus answering a question posed by Flach [9].

In this paper, we concentrate on algebras of dihedral type of polynomial growth. Algebras of dihedral type are certain symmetric algebras which played an important role in Erdmann's classification of all tame blocks of group algebras with dihedral defect groups up to Morita equivalence (see [10]). A k -algebra is said to be of polynomial growth if its indecomposable modules of any given

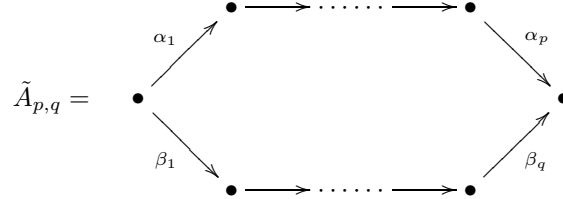
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k -dimension d can be parameterized using only a finite number $\mu(d)$ of one-parameter families and if there is a natural number m such that $\mu(d) \leq d^m$ for all $d \geq 1$.

Erdmann and Skowroński classified in [11, Sect. 4] all basic algebras of dihedral type of polynomial growth, by providing the quiver and relations of each such algebra. They showed that there are precisely 8 possible quivers, each of which having at most 3 vertices, and 12 Morita equivalence classes of algebras. It follows that the stable Auslander-Reiten quiver of each such algebra Λ has precisely one non-periodic component of the form $\mathbb{Z}\tilde{A}_{p,q}$ where $(p, q) \in \{(1, 1), (3, 1), (3, 3)\}$, depending on whether the quiver of Λ has 1, 2, or 3 vertices. Moreover, there are up to two 3-tubes and infinitely many 1-tubes. Here $\tilde{A}_{p,q}$ denotes the quiver



A summary of our main results is as follows; for more precise statements, see Propositions 3.1, 3.2 and 3.3. Note that Ω stands for the syzygy functor (see e.g. [1, pp. 124–126]).

Theorem 1.1. *Let $\Lambda = kQ/I$ be an algebra of dihedral type of polynomial growth, and let \mathfrak{C} be a connected component of the stable Auslander-Reiten quiver of Λ .*

- (i) *If \mathfrak{C} is a non-periodic component, then \mathfrak{C} is stable under Ω and the stable endomorphism ring of every module V belonging to \mathfrak{C} is isomorphic to k . If \mathfrak{C} is of the form $\mathbb{Z}\tilde{A}_{1,1}$, then there is precisely one Ω -orbit of Λ -modules V in \mathfrak{C} and $R(\Lambda, V) \cong \Lambda$. If \mathfrak{C} is of the form $\mathbb{Z}\tilde{A}_{3,1}$, then there are precisely two Ω -orbits of Λ -modules V in \mathfrak{C} and $R(\Lambda, V) \cong k[[t]]/(t^2)$. If \mathfrak{C} is of the form $\mathbb{Z}\tilde{A}_{3,3}$, then there are precisely three Ω -orbits of Λ -modules V in \mathfrak{C} and $R(\Lambda, V)$ is isomorphic to either k or $k[[t]]/(t^2)$.*
- (ii) *Suppose \mathfrak{C} is a 3-tube. If $\Omega(\mathfrak{C}) = \mathfrak{C}$, then the only modules in \mathfrak{C} whose stable endomorphism rings are isomorphic to k are the modules V at the boundary of \mathfrak{C} and $R(\Lambda, V) \cong k$. If $\Omega(\mathfrak{C}) \neq \mathfrak{C}$, then there are precisely three Ω -orbits of modules V in $\mathfrak{C} \cup \Omega(\mathfrak{C})$ whose stable endomorphism rings are isomorphic to k and $R(\Lambda, V)$ is isomorphic to either k or $k[[t]]$.*
- (iii) *Suppose \mathfrak{C} is a 1-tube. If $\Omega(\mathfrak{C}) = \mathfrak{C}$, then no module in \mathfrak{C} has a stable endomorphism ring isomorphic to k . If $\Omega(\mathfrak{C}) \neq \mathfrak{C}$, then the only module in \mathfrak{C} whose stable endomorphism ring is isomorphic to k is the module V at the boundary of \mathfrak{C} and $R(\Lambda, V) \cong k[[t]]$.*

The main steps to prove Theorem 1.1 are as follows: We use that for each algebra Λ of dihedral type of polynomial growth, $\Lambda/\text{soc}(\Lambda)$ is a string algebra, which means that all non-projective indecomposable Λ -modules are given combinatorially by string and band modules (see [8]). We use the description of the Λ -module homomorphisms between string and band modules given in [12] to find all Λ -modules V whose stable endomorphism rings are isomorphic to k . Since by [6] the universal deformation ring of V is preserved by the syzygy functor Ω , it turns out that we only have to consider finitely many isomorphism classes of string modules V whose stable endomorphism rings are isomorphic to k . Moreover, there is precisely one band attached to each Λ , which means that we only have to determine for which $\mu \in k^*$ the stable endomorphism ring of the band module V which is associated to μ and which lies at the boundary of its 1-tube is isomorphic to k . We then compute the k -dimension of $\text{Ext}_\Lambda^1(V, V)$ for each obtained string or band module V and use this to determine the isomorphism type of the universal deformation ring $R(\Lambda, V)$.

Let G be a finite group. In [11, Thm. 4.1], the blocks B of kG which are representation infinite of polynomial growth were determined. It was shown that B is such a block if and only if $\text{char}(k) = 2$ and the defect groups of B are Klein four groups. In particular, B is an algebra of dihedral type of polynomial growth. The determination of the universal deformation rings in Theorem 1.1 enables us to give a characterization of representation infinite blocks of polynomial growth in terms of universal

deformation rings (see Corollary 3.4). Note that the full universal deformation rings of modules for blocks with Klein four defect groups were determined in [2].

The paper is organized as follows. In §2, we recall the definitions of deformations and deformation rings. In §3, we describe the quivers and relations of all basic algebras Λ of dihedral type of polynomial growth provided in [11, Sect. 4] and prove Theorem 1.1. In §4, we give a brief introduction to string and band modules as given in [8].

Part of this paper constitutes the Ph.D. thesis of the second author under the supervision of the first author [13].

2. VERSAL AND UNIVERSAL DEFORMATION RINGS

In this section, we give a brief introduction to versal and universal deformation rings and deformations. For more background material, we refer the reader to [6].

Let k be a field of arbitrary characteristic. Let $\hat{\mathcal{C}}$ be the category of all complete local commutative Noetherian k -algebras with residue field k . The morphisms in $\hat{\mathcal{C}}$ are continuous k -algebra homomorphisms which induce the identity map on k .

Suppose Λ is a finite dimensional k -algebra and V is a finitely generated Λ -module. A lift of V over an object R in $\hat{\mathcal{C}}$ is a finitely generated $R \otimes_k \Lambda$ -module M which is free over R together with a Λ -module isomorphism $\phi : k \otimes_R M \rightarrow V$. Two lifts (M, ϕ) and (M', ϕ') of V over R are isomorphic if there exists an $R \otimes_k \Lambda$ -module isomorphism $f : M \rightarrow M'$ such that $\phi' \circ (k \otimes_R f) = \phi$. The isomorphism class of a lift (M, ϕ) of V over R is denoted by $[M, \phi]$ and called a deformation of V over R . We denote the set of all such deformations of V over R by $\text{Def}_\Lambda(V, R)$. The deformation functor

$$\hat{F}_V : \hat{\mathcal{C}} \rightarrow \text{Sets}$$

is the covariant functor which sends an object R in $\hat{\mathcal{C}}$ to $\text{Def}_\Lambda(V, R)$ and a morphism $\alpha : R \rightarrow R'$ in $\hat{\mathcal{C}}$ to the map $\text{Def}_\Lambda(V, R) \rightarrow \text{Def}_\Lambda(V, R')$ defined by $[M, \phi] \mapsto [R' \otimes_{R, \alpha} M, \phi_\alpha]$, where $\phi_\alpha = \phi$ after identifying $k \otimes_{R'} (R' \otimes_{R, \alpha} M)$ with $k \otimes_R M$.

Suppose there exists an object $R(\Lambda, V)$ in $\hat{\mathcal{C}}$ and a deformation $[U(\Lambda, V), \phi_U]$ of V over $R(\Lambda, V)$ with the following property: For each R in $\hat{\mathcal{C}}$ and for each lift (M, ϕ) of V over R there exists a morphism $\alpha : R(\Lambda, V) \rightarrow R$ in $\hat{\mathcal{C}}$ such that $\hat{F}_V(\alpha)([U(\Lambda, V), \phi_U]) = [M, \phi]$, and moreover α is unique if R is the ring of dual numbers $k[\epsilon]/(\epsilon^2)$. Then $R(\Lambda, V)$ is called the versal deformation ring of V and $[U(\Lambda, V), \phi_U]$ is called the versal deformation of V . If the morphism α is unique for all R and all lifts (M, ϕ) of V over R , then $R(\Lambda, V)$ is called the universal deformation ring of V and $[U(\Lambda, V), \phi_U]$ is called the universal deformation of V . In other words, $R(\Lambda, V)$ is universal if and only if $R(\Lambda, V)$ represents the functor \hat{F}_V in the sense that \hat{F}_V is naturally isomorphic to the Hom functor $\text{Hom}_{\hat{\mathcal{C}}}(R(\Lambda, V), -)$.

Note that the above definition of deformations can be weakened as follows. Given a lift (M, ϕ) of V over a ring R in $\hat{\mathcal{C}}$, define the corresponding weak deformation to be the isomorphism class of M as an $R \otimes_k \Lambda$ -module, without taking into account the specific isomorphism $\phi : k \otimes_R M \rightarrow V$. In general, a weak deformation of V over R identifies more lifts than a deformation of V over R that respects the isomorphism ϕ of a representative (M, ϕ) . However, if Λ is self-injective and the stable endomorphism ring $\underline{\text{End}}_\Lambda(V)$ is isomorphic to k , these two definitions of deformations coincide (see [6, Thm. 2.6]).

It is straightforward to check that every finitely generated Λ -module V has a versal deformation ring and that this versal deformation ring is universal if the endomorphism ring $\text{End}_\Lambda(V)$ is isomorphic to k (see [6, Prop. 2.1]). Moreover, Morita equivalences preserve versal deformation rings (see [6, Prop. 2.5]). If Λ is self-injective, we obtain the following result, where Ω denotes the syzygy functor (see e.g. [1, pp. 124–126]).

Theorem 2.1. ([6, Thm. 2.6]) *Let Λ be a finite dimensional self-injective k -algebra, and suppose V is a finitely generated Λ -module whose stable endomorphism ring $\underline{\text{End}}_\Lambda(V)$ is isomorphic to k .*

- (i) *The module V has a universal deformation ring $R(\Lambda, V)$.*

- (ii) If P is a finitely generated projective Λ -module, then $\underline{\text{End}}_{\Lambda}(V \oplus P) \cong k$ and $R(\Lambda, V) \cong R(\Lambda, V \oplus P)$.
- (iii) If Λ is moreover a Frobenius algebra, then $\underline{\text{End}}_{\Lambda}(\Omega(V)) \cong k$ and $R(\Lambda, V) \cong R(\Lambda, \Omega(V))$.

3. ALGEBRAS OF DIHEDRAL TYPE OF POLYNOMIAL GROWTH

In this section we consider all algebras of dihedral type which are of polynomial growth. Let k be an algebraically closed field of arbitrary characteristic. Algebras of dihedral type are certain symmetric algebras which play an important role in Erdmann's classification of all tame blocks of group algebras with dihedral defect groups up to Morita equivalence (see [10]). Since by [7], these tame blocks have at most three isomorphism classes of simple modules, this is also the case for the algebras of dihedral type. However, the class of algebras of dihedral type strictly contains the Morita equivalence classes of these tame blocks. A k -algebra is said to be of polynomial growth if it is tame, i.e. its indecomposable modules of any given k -dimension d can (up to finitely many exceptions) be parameterized using only a finite number $\mu(d)$ of one-parameter families, and if there is a natural number m such that $\mu(d) \leq d^m$ for all $d \geq 1$. Erdmann and Skowroński classified in [11, Sect. 4] all algebras of dihedral type of polynomial growth up to Morita equivalence. In particular, they showed in [11, Thm. 4.1] that the only tame blocks with dihedral defect groups which are of polynomial growth have Klein four defect groups. The latter fall into three distinct Morita equivalence classes of algebras with either exactly one or exactly three isomorphism classes of simple modules.

Let $\Lambda = kQ/I$ be an algebra of dihedral type of polynomial growth. Then Q is one of 8 possible quivers listed in Figure 1. For each Q , Erdmann and Skowroński provide either one or two ideals I of kQ to obtain a complete list of Morita equivalence classes of algebras of dihedral type of polynomial growth. These ideals I are listed in Figure 2.

For each algebra Λ in Figure 2, $\Lambda/\text{soc}(\Lambda)$ is a string algebra. In particular, all indecomposable non-projective Λ -modules are given combinatorially by string and band modules (see [8]). Moreover, the Λ -module homomorphisms between string and band modules have been explicitly described in [12]. For the convenience of the reader, we give a brief introduction to string and band modules in §4.

For each Λ , there exists a unique pair $(p, q) \in \{(1, 1), (3, 1), (3, 3)\}$ and a unique band B such that the stable Auslander-Reiten quiver of Λ consists of a non-periodic component of the form $\mathbb{Z}\tilde{A}_{p,q}$, one p -tube and one q -tube consisting of string modules, and for each $\mu \in k^*$ a 1-tube consisting of band modules with $M(B, \mu, 1)$ lying at the boundary of this 1-tube.

In the following subsections, we find for each Λ in Figure 2 all indecomposable modules whose stable endomorphism rings are isomorphic to k and we determine for each such module its universal deformation ring. We organize the modules according to the components of the stable Auslander-Reiten quiver of Λ to which they belong.

3.1. Non-periodic components. In this subsection, we show that the stable endomorphism ring of every module which belongs to the non-periodic component of the stable Auslander-Reiten quiver is isomorphic to k and determine the universal deformation ring of each of these modules.

Proposition 3.1. *Let $\Lambda = kQ/I$ be a basic algebra of dihedral type of polynomial growth, and let \mathfrak{C} be the non-periodic component of the stable Auslander-Reiten quiver of Λ . Then the stable endomorphism ring of every module belonging to \mathfrak{C} is isomorphic to k .*

- (i) If $\Lambda = D(1)_c$ for $c \in \{0, 1\}$, then there is precisely one Ω -orbit of modules in \mathfrak{C} and $R(\Lambda, V) \cong \Lambda$ for every module V belonging to \mathfrak{C} .
- (ii) If $\Lambda = D(2\mathcal{A})_c$ for $c \in \{0, 1\}$, then there are precisely two Ω -orbits of modules in \mathfrak{C} and $R(\Lambda, V) \cong k[[t]]/(t^2)$ for every module V belonging to \mathfrak{C} .
- (iii) If $\Lambda \in \{D(3\mathcal{A})_1, D(3\mathcal{K})\}$, then there are precisely three Ω -orbits of modules in \mathfrak{C} and $R(\Lambda, V) \cong k$ for every module V belonging to \mathfrak{C} .

FIGURE 1. Quivers of algebras of dihedral type of polynomial growth.

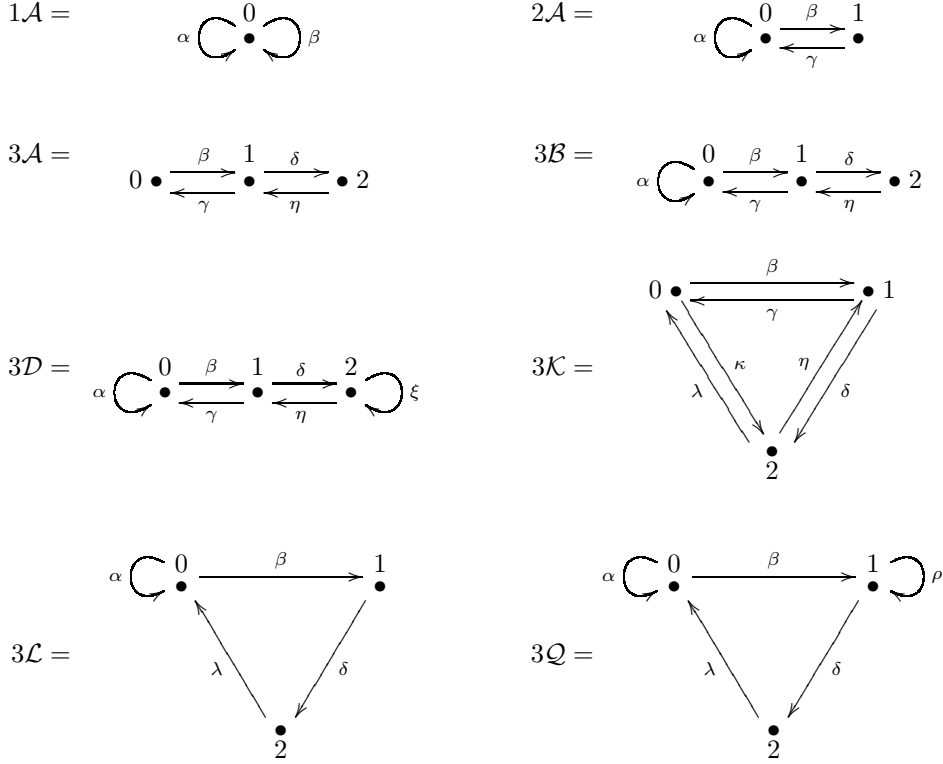


FIGURE 2. Algebras of dihedral type of polynomial growth.

$$\begin{aligned}
D(1)_0 &= k[1\mathcal{A}]/\langle \alpha^2, \beta^2, \alpha\beta - \beta\alpha \rangle, \\
D(1)_1 &= k[1\mathcal{A}]/\langle \alpha^2, \beta^2 - \alpha\beta, \alpha\beta - \beta\alpha \rangle \quad \text{and } \text{char}(k) = 2, \\
D(2\mathcal{A})_0 &= k[2\mathcal{A}]/\langle \alpha^2, \beta\gamma, \alpha\gamma\beta - \gamma\beta\alpha \rangle, \\
D(2\mathcal{A})_1 &= k[2\mathcal{A}]/\langle \alpha^2 - \alpha\gamma\beta, \beta\gamma, \alpha\gamma\beta - \gamma\beta\alpha \rangle \quad \text{and } \text{char}(k) = 2, \\
D(3\mathcal{A})_1 &= k[3\mathcal{A}]/\langle \gamma\beta, \delta\eta, \beta\gamma\eta\delta - \eta\delta\beta\gamma \rangle, \\
D(3\mathcal{A})_2 &= k[3\mathcal{A}]/\langle \gamma\eta, \delta\beta, (\beta\gamma)^2 - (\eta\delta)^2 \rangle, \\
D(3\mathcal{B})_{2,1} &= k[3\mathcal{B}]/\langle \alpha\gamma, \beta\alpha, \gamma\eta, \delta\beta, \alpha^2 - \gamma\beta, \beta\gamma - (\eta\delta)^2 \rangle, \\
D(3\mathcal{B})_{2,2} &= k[3\mathcal{B}]/\langle \alpha\gamma, \beta\alpha, \gamma\eta, \delta\beta, \alpha^2 - (\gamma\beta)^2, (\beta\gamma)^2 - \eta\delta \rangle, \\
D(3\mathcal{D})_2 &= k[3\mathcal{D}]/\langle \alpha\gamma, \beta\alpha, \gamma\eta, \delta\beta, \eta\xi, \xi\delta, \alpha^2 - \gamma\beta, \beta\gamma - \eta\delta, \delta\eta - \xi^2 \rangle, \\
D(3\mathcal{K}) &= k[3\mathcal{K}]/\langle \beta\lambda, \gamma\eta, \delta\beta, \eta\kappa, \kappa\gamma, \lambda\delta, \beta\gamma - \eta\delta, \gamma\beta - \lambda\kappa, \delta\eta - \kappa\lambda \rangle, \\
D(3\mathcal{L}) &= k[3\mathcal{L}]/\langle \alpha\lambda, \beta\alpha, \alpha^2 - (\lambda\delta\beta)^2, \delta(\beta\lambda\delta)^2 \rangle, \\
D(3\mathcal{Q}) &= k[3\mathcal{Q}]/\langle \alpha\lambda, \beta\alpha, \delta\rho, \rho\beta, \alpha^2 - \lambda\delta\beta, \beta\lambda\delta - \rho^2 \rangle.
\end{aligned}$$

- (iv) If $\Lambda \in \{D(3\mathcal{A})_2, D(3\mathcal{B})_{2,1}, D(3\mathcal{B})_{2,2}, D(3\mathcal{D})_2, D(3\mathcal{L}), D(3\mathcal{Q})\}$, then there are precisely three Ω -orbits of modules in \mathfrak{C} and $R(\Lambda, V) \cong k$ for V in precisely one of these Ω -orbits and $R(\Lambda, V) \cong k[[t]]/(t^2)$ for V in the remaining two Ω -orbits.

Proof. Suppose first that Q has a unique vertex, i.e. $\Lambda = D(1)_c$ for $c \in \{0, 1\}$, where $c = 1$ only occurs when $\text{char}(k) = 2$. Then \mathfrak{C} is of the form $\mathbb{Z}\tilde{A}_{1,1}$ and there is precisely one Ω -orbit of Λ -modules in \mathfrak{C} represented by the simple module S_0 corresponding to the unique vertex in Q . To prove part (i), we can by Theorem 2.1 restrict to the case when $V = S_0$. Note that Λ is a ring in $\hat{\mathcal{C}}$ and that $U = \Lambda$ defines a lift of S_0 over Λ , where a simple tensor $a \otimes b \in \Lambda \otimes_k \Lambda$ acts on U as multiplication by ab . Let R be an arbitrary ring in $\hat{\mathcal{C}}$ and let M be a lift of S_0 over R . Then the Λ -action on $M = R$ is given by a continuous k -algebra homomorphism $\tau_M : \Lambda \rightarrow R$ which induces the identity on the residue field k , i.e. τ_M is a morphism in $\hat{\mathcal{C}}$. Moreover, since M is free of rank 1 over R , $\text{Aut}_{R \otimes_k \Lambda}(M) \cong \text{Aut}_R(M) \cong \text{GL}_1(R)$, which implies that τ_M is unique. Since $M = R \otimes_{\Lambda, \tau_M} U$, it follows that U defines a universal lift of S_0 over Λ . In other words, $R(\Lambda, S_0) \cong \Lambda$, which proves part (i).

Next suppose that Q has two vertices, i.e. $\Lambda = D(2A)_c$ for $c \in \{0, 1\}$, where $c = 1$ only occurs when $\text{char}(k) = 2$. Then \mathfrak{C} is of the form $\mathbb{Z}\tilde{A}_{3,1}$ and there are precisely two Ω -orbits of Λ -modules in \mathfrak{C} represented by the simple module S_0 and the string module $M(\beta)$. To prove part (ii), we can by Theorem 2.1 restrict to the case when $V \in \{S_0, M(\beta)\}$. We have $\text{End}_\Lambda(V) \cong k \cong \text{Ext}_\Lambda^1(V, V)$, which implies that $R(\Lambda, V)$ is isomorphic to a quotient algebra of $k[[t]]$. There exists a non-split short exact sequence of Λ -modules

$$0 \rightarrow V \xrightarrow{\iota} M \xrightarrow{\tau} V \rightarrow 0$$

where $M = M(\alpha)$ (resp. $M = M(\beta\alpha^{-1}\beta^{-1})$) if $V = S_0$ (resp. $V = M(\beta)$). This means that M defines a non-trivial lift of V over $k[[t]]/(t^2)$ where the action of t is given by $\iota \circ \tau$. Hence there exists a unique surjective k -algebra homomorphism $\psi : R(\Lambda, V) \rightarrow k[[t]]/(t^2)$ in $\hat{\mathcal{C}}$ corresponding to the deformation defined by M . We need to show that ψ is an isomorphism. Suppose this is false. Then there exists a surjective k -algebra homomorphism $\psi_0 : R(\Lambda, V) \rightarrow k[[t]]/(t^3)$ in $\hat{\mathcal{C}}$ such that $\pi \circ \psi_0 = \psi$ where $\pi : k[[t]]/(t^3) \rightarrow k[[t]]/(t^2)$ is the natural projection. Let M_0 be a $k[[t]]/(t^3) \otimes_k \Lambda$ -module which defines a lift of V over $k[[t]]/(t^3)$ corresponding to ψ_0 . Because $M_0/t^2 M_0 \cong M$ and $t^2 M_0 \cong V$, we obtain a non-split short exact sequence of $k[[t]]/(t^3) \otimes_k \Lambda$ -modules

$$0 \rightarrow V \rightarrow M_0 \rightarrow M \rightarrow 0.$$

Since $\text{Ext}_\Lambda^1(M, V) = 0$, this sequence splits as a sequence of Λ -modules. Hence $M_0 = V \oplus M$ as Λ -modules. Writing elements of M_0 as (v, x) where $v \in V$ and $x \in M$, the action of t on M_0 is given by $t(v, x) = (\sigma(x), tx)$, where $\sigma : M \rightarrow V$ is a surjective Λ -module homomorphism. Since for each such σ its kernel is equal to tM , it follows that $t^2(v, x) = (\sigma(tx), t^2x) = (0, 0)$ for all $v \in V$ and $x \in M$. But this is a contradiction to $t^2 M_0 \cong V$. Thus ψ is a k -algebra isomorphism and $R(\Lambda, V) \cong k[[t]]/(t^2)$, which proves part (ii).

Next suppose Q has precisely three vertices. Then \mathfrak{C} is of the form $\mathbb{Z}\tilde{A}_{3,3}$ and there are precisely three Ω -orbits of Λ -modules in \mathfrak{C} represented by V_0, V_1, V_2 as follows:

- If $\Lambda \in \{D(3A)_1, D(3A)_2\}$, then $V_0 = S_1, V_1 = M(\beta), V_2 = M(\eta)$.
- If $\Lambda = D(3B)_{2,1}$, then $V_0 = S_1, V_1 = S_0, V_2 = M(\eta)$.
- If $\Lambda = D(3B)_{2,2}$, then $V_0 = M(\gamma\delta^{-1}), V_1 = S_0, V_2 = M(\gamma)$.
- If $\Lambda \in \{D(3D)_2, D(3K)\}$, then $V_0 = S_1, V_1 = S_0, V_2 = S_2$.
- If $\Lambda = D(3L)$, then $V_0 = M(\beta), V_1 = S_0, V_2 = M(\delta\beta)$.
- If $\Lambda = D(3Q)$, then $V_0 = M(\delta), V_1 = S_0, V_2 = S_1$.

In all cases, $\text{End}_\Lambda(V_i) \cong k$ for $i \in \{0, 1, 2\}$ and $\text{Ext}_\Lambda^1(V_0, V_0) = 0$. Hence $R(\Lambda, V_0) \cong k$.

If $\Lambda \in \{D(3A)_1, D(3K)\}$, then $\text{Ext}_\Lambda^1(V_i, V_i) = 0$ also when $i \in \{1, 2\}$, implying $R(\Lambda, V_i) \cong k$ for all $i \in \{0, 1, 2\}$, which proves part (iii).

Finally, suppose $\Lambda \in \{D(3A)_2, D(3B)_{2,1}, D(3B)_{2,2}, D(3D)_2, D(3L), D(3Q)\}$, and let $i \in \{1, 2\}$. Then $\text{Ext}_\Lambda^1(V_i, V_i) \cong k$, which implies that $R(\Lambda, V_i)$ is isomorphic to a quotient algebra of $k[[t]]$. To prove that $R(\Lambda, V_i) \cong k[[t]]/(t^2)$, we first show that V_i has a non-trivial lift M_i over $k[[t]]/(t^2)$. We define M_i as follows:

- If $\Lambda = D(3A)_2$, let $M_1 = M(\beta\gamma\beta)$ and $M_2 = M(\eta\delta\eta)$.

- If $\Lambda = D(3\mathcal{B})_{2,1}$, let $M_1 = M(\alpha)$ and $M_2 = M(\eta\delta\eta)$.
- If $\Lambda = D(3\mathcal{B})_{2,2}$, let $M_1 = M(\alpha)$ and $M_2 = M(\gamma\beta\gamma)$.
- If $\Lambda = D(3\mathcal{D})_2$, let $M_1 = M(\alpha)$ and $M_2 = M(\xi)$.
- If $\Lambda = D(3\mathcal{L})$, let $M_1 = M(\alpha)$ and $M_2 = M(\delta\beta\lambda\delta\beta)$.
- If $\Lambda = D(3\mathcal{Q})$, let $M_1 = M(\alpha)$ and $M_2 = M(\rho)$.

Let $i \in \{1, 2\}$. In all cases, there exists a non-split short exact sequence of Λ -modules

$$0 \rightarrow V_i \xrightarrow{\iota_i} M_i \xrightarrow{\tau_i} V_i \rightarrow 0$$

which means that M_i defines a non-trivial lift of V_i over $k[[t]]/(t^2)$ where the action of t is given by $\iota_i \circ \tau_i$. Since $\text{Ext}_\Lambda^1(M_i, V_i) = 0$ and the only surjective Λ -module homomorphisms $\sigma_i : M_i \rightarrow V_i$ have kernel equal to tM_i , we can argue similarly to the case $D(2\mathcal{A})_c$ to show that $R(\Lambda, V_i) \cong k[[t]]/(t^2)$ for $i \in \{1, 2\}$, proving part (iv). \square

3.2. 3-tubes. In this subsection, we find all modules belonging to 3-tubes of the stable Auslander-Reiten quiver whose stable endomorphism rings are isomorphic to k and determine their universal deformation rings.

Proposition 3.2. *Let $\Lambda = kQ/I$ be a basic algebra of dihedral type of polynomial growth, let \mathfrak{T} be a 3-tube of the stable Auslander-Reiten quiver of Λ consisting of string modules, and let T_0 be a module belonging to the boundary of \mathfrak{T} .*

- (i) *If $\Lambda = D(1)_c$ for $c \in \{0, 1\}$, then \mathfrak{T} does not exist.*
- (ii) *If $\Lambda = D(2\mathcal{A})_c$ for $c \in \{0, 1\}$, then \mathfrak{T} is unique and stable under Ω . There is precisely one Ω -orbit of modules in \mathfrak{T} whose stable endomorphism rings are isomorphic to k represented by T_0 , and $R(\Lambda, T_0) \cong k$.*
- (iii) *If $\Lambda \in \{D(3\mathcal{A})_1, D(3\mathcal{K})\}$, then \mathfrak{T} is one of two 3-tubes and \mathfrak{T} is stable under Ω . There is precisely one Ω -orbit of modules in \mathfrak{T} whose stable endomorphism rings are isomorphic to k represented by T_0 , and $R(\Lambda, T_0) \cong k$.*
- (iv) *If $\Lambda \in \{D(3\mathcal{A})_2, D(3\mathcal{B})_{2,1}, D(3\mathcal{B})_{2,2}, D(3\mathcal{D})_2, D(3\mathcal{L}), D(3\mathcal{Q})\}$, then \mathfrak{T} is one of two 3-tubes and Ω interchanges these two 3-tubes. There are precisely three Ω -orbits of modules in $\mathfrak{T} \cup \Omega(\mathfrak{T})$ whose stable endomorphism rings are isomorphic to k represented by T_0 , by a successor T_1 of T_0 , and by a successor T_2 of T_1 which does not lie in the Ω -orbit of T_0 . Moreover, $R(\Lambda, T_0) \cong k \cong R(\Lambda, T_1)$ and $R(\Lambda, T_2) \cong k[[t]]$.*

Proof. If $\Lambda = D(1)_c$ then there are precisely two maximal directed strings, which means that the stable Auslander-Reiten quiver of Λ does not contain any 3-tubes.

Suppose now that $\Lambda = D(2\mathcal{A})_c$ for $c \in \{0, 1\}$, where $c = 1$ only occurs when $\text{char}(k) = 2$. Then \mathfrak{T} is unique and hence stable under Ω . Using hooks and cohooks (see [8, pp. 166–174]), we see that the Ω -orbits of all Λ -modules in \mathfrak{T} are represented by S_1 , which lies at the boundary of \mathfrak{T} , and by

$$\begin{aligned} T_{1,j} &= M((\alpha^{-1}\gamma\beta)^{j-1}\alpha^{-1}), \\ T_{2,j} &= M((\alpha^{-1}\gamma\beta)^{j-1}\alpha^{-1}\gamma), \\ T_{3,j} &= M((\alpha^{-1}\gamma\beta)^j\alpha^{-1}\beta^{-1}) \end{aligned}$$

for all $j \geq 1$. For $i \in \{1, 2, 3\}$ and $j \geq 1$, there exists an endomorphism of $T_{i,j}$ whose image is isomorphic to S_0 and which does not factor through a projective Λ -module. Hence, the only modules in \mathfrak{T} whose stable endomorphism rings are isomorphic to k lie in the Ω -orbit of S_1 . Since $\text{Ext}_\Lambda^1(S_1, S_1) = 0$, we obtain that $R(\Lambda, T_0) \cong k$ for all T_0 at the boundary of \mathfrak{T} , which proves part (ii).

Next suppose Λ has precisely three isomorphism classes of simple modules. Then \mathfrak{T} is one of two 3-tubes.

If $\Lambda = D(3\mathcal{A})_1$ (resp. $\Lambda = D(3\mathcal{K})$), then \mathfrak{T} is stable under Ω and the Ω -orbit of the modules at the boundary of \mathfrak{T} is represented by $T_0 = S_1$ or $T_0 = S_2$ (resp. by $T_0 = M(\gamma)$ or $T_0 = M(\lambda)$). Arguing similarly to the case $D(2\mathcal{A})_c$, we see that the only modules in \mathfrak{T} whose stable endomorphism rings

are isomorphic to k lie in the Ω -orbit of T_0 . Since $\text{Ext}_\Lambda^1(T_0, T_0) = 0$, we obtain that $R(\Lambda, T_0) \cong k$, which proves part (iii).

If $\Lambda \in \{D(3\mathcal{A})_2, D(3\mathcal{B})_{2,1}, D(3\mathcal{B})_{2,2}, D(3\mathcal{D})_2, D(3\mathcal{L}), D(3\mathcal{Q})\}$, then $\Omega(\mathfrak{T}) \neq \mathfrak{T}$. The representatives T_0, T_1, T_2 from the statement of part (iv) can be taken to be as follows:

- If $\Lambda = D(3\mathcal{A})_2$, then $T_0 = S_0, T_1 = M(\gamma\delta^{-1}\eta^{-1}\delta^{-1}), T_2 = M(\gamma\delta^{-1}\eta^{-1})$.
- If $\Lambda = D(3\mathcal{B})_{2,1}$, then $T_0 = S_2, T_1 = M(\delta\gamma^{-1}), T_2 = M(\delta\gamma^{-1}\alpha\beta^{-1})$.
- If $\Lambda = D(3\mathcal{B})_{2,2}$, then $T_0 = M(\delta^{-1}), T_1 = S_1, T_2 = M(\beta\alpha^{-1})$.
- If $\Lambda = D(3\mathcal{D})_2$, then $T_0 = M(\gamma^{-1}), T_1 = M(\gamma^{-1}\alpha\beta^{-1}), T_2 = M(\gamma^{-1}\alpha\beta^{-1}\eta\xi^{-1})$.
- If $\Lambda = D(3\mathcal{L})$, then $T_0 = S_2, T_1 = M(\delta), T_2 = M(\delta\beta\alpha^{-1})$.
- If $\Lambda = D(3\mathcal{Q})$, then $T_0 = S_2, T_1 = M(\delta\rho^{-1}), T_2 = M(\delta\rho^{-1}\beta\alpha^{-1})$.

Arguing similarly to the case $D(2\mathcal{A})_c$, we see that the only modules in \mathfrak{T} whose stable endomorphism rings are isomorphic to k lie in the Ω -orbits of T_0, T_1, T_2 . If $i \in \{0, 1\}$ then $\text{Ext}_\Lambda^1(T_i, T_i) = 0$ and hence $R(\Lambda, T_i) \cong k$. On the other hand, $\text{Ext}_\Lambda^1(T_2, T_2) = k$, which implies that $R(\Lambda, T_2)$ is isomorphic to a quotient algebra of $k[[t]]$. To prove that $R(\Lambda, T_2) \cong k[[t]]$, it is therefore enough to prove that T_2 has a lift L_2 over $k[[t]]$ such that L_2/t^2L_2 defines a non-trivial lift of T_2 over $k[[t]]/(t^2)$. We define L_2 as follows:

- If $\Lambda = D(3\mathcal{A})_2$, let L_2 be the free $k[[t]]$ -module of rank 4 with an ordered basis $\{B_0, B_1, B_2, B_3\}$ and define a Λ -module structure on L_2 by letting (the image of) each vertex (resp. arrow) c in Q act on $\{B_0, B_1, B_2, B_3\}$ as the following 4×4 matrix X_c : $X_{e_0} = E_{00}, X_{e_1} = E_{11} + E_{33}, X_{e_2} = E_{22}, X_\beta = tE_{30}, X_\gamma = E_{01}, X_\delta = E_{21}$ and $X_\eta = E_{32}$. Here E_{ji} denotes the 4×4 matrix which sends B_i to B_j and all other basis elements to 0.
- If $\Lambda = D(3\mathcal{B})_{2,1}$, let L_2 be the free $k[[t]]$ -module of rank 5 with an ordered basis $\{B_0, \dots, B_4\}$ and define a Λ -module structure on L_2 by letting (the image of) each vertex (resp. arrow) c in Q act on $\{B_0, \dots, B_4\}$ as the following 5×5 matrix X_c : $X_{e_0} = E_{22} + E_{33}, X_{e_1} = E_{11} + E_{44}, X_{e_2} = E_{00}, X_\alpha = E_{23}, X_\beta = E_{43}, X_\gamma = E_{21}, X_\delta = E_{01}$ and $X_\eta = tE_{40}$.
- If $\Lambda = D(3\mathcal{B})_{2,2}$, let L_2 be the free $k[[t]]$ -module of rank 3 with an ordered basis $\{B_0, B_1, B_2\}$ and let (the image of) each vertex (resp. arrow) c in Q act on $\{B_0, B_1, B_2\}$ as the following 3×3 matrix X_c : $X_{e_0} = E_{11} + E_{22}, X_{e_1} = E_{00}, X_{e_2} = 0, X_\alpha = E_{21}, X_\beta = E_{01}, X_\gamma = tE_{20}, X_\delta = 0$ and $X_\eta = 0$.
- If $\Lambda = D(3\mathcal{D})_2$, let L_2 be the free $k[[t]]$ -module of rank 6 with an ordered basis $\{B_0, \dots, B_5\}$ and let (the image of) each vertex (resp. arrow) c in Q act on $\{B_0, \dots, B_5\}$ as the following 6×6 matrix X_c : $X_{e_0} = E_{11} + E_{22}, X_{e_1} = E_{00} + E_{33}, X_{e_2} = E_{44} + E_{55}, X_\alpha = E_{12}, X_\beta = E_{32}, X_\gamma = E_{10}, X_\delta = tE_{50}, X_\eta = E_{34}$ and $X_\xi = E_{54}$.
- If $\Lambda = D(3\mathcal{L})$, let L_2 be the free $k[[t]]$ -module of rank 4 with an ordered basis $\{B_0, \dots, B_3\}$ and let (the image of) each vertex (resp. arrow) c in Q act on $\{B_0, \dots, B_3\}$ as the following 4×4 matrix X_c : $X_{e_0} = E_{22} + E_{33}, X_{e_1} = E_{11}, X_{e_2} = E_{00}, X_\alpha = E_{32}, X_\beta = E_{12}, X_\delta = E_{01}$ and $X_\lambda = tE_{30}$.
- If $\Lambda = D(3\mathcal{Q})$, let L_2 be the free $k[[t]]$ -module of rank 5 with an ordered basis $\{B_0, \dots, B_4\}$ and let (the image of) each vertex (resp. arrow) c in Q act on $\{B_0, \dots, B_4\}$ as the following 5×5 matrix X_c : $X_{e_0} = E_{33} + E_{44}, X_{e_1} = E_{11} + E_{22}, X_{e_2} = E_{00}, X_\alpha = E_{43}, X_\beta = E_{23}, X_\delta = E_{01}, X_\lambda = tE_{40}$ and $X_\rho = E_{21}$.

In all cases, L_2 is a $k[[t]] \otimes_k \Lambda$ -module which is free as a $k[[t]]$ -module, $L_2/tL_2 \cong T_2$ as Λ -modules, and L_2/t^2L_2 defines a non-trivial lift of T_2 over $k[[t]]/(t^2)$. This implies that $R(\Lambda, T_2) \cong k[[t]]$, proving part (iv). \square

3.3. 1-tubes. In this subsection, we find all modules belonging to 1-tubes of the stable Auslander-Reiten quiver whose stable endomorphism rings are isomorphic to k and determine their universal deformation rings.

For each Λ in Figure 2, there exists a unique band B . For each $\mu \in k^*$, the band module $M(B, \mu, 1)$ lies at the boundary of its 1-tube. If $\Lambda = D(1)_c$ then $B = \beta\alpha^{-1}$, and we allow μ to lie in $k \cup \{\infty\}$ by defining $M(\beta\alpha^{-1}, 0, m) = M(\alpha^{-1}(\beta\alpha^{-1})^{m-1})$ and $M(\beta\alpha^{-1}, \infty, m) = M((\beta\alpha^{-1})^{m-1}\beta)$.

for all $m \in \mathbb{Z}^+$. If $\Lambda = D(2\mathcal{A})_c$ then $B = \alpha\beta^{-1}\gamma^{-1}$, and we allow μ to lie in k by defining $M(\alpha\beta^{-1}\gamma^{-1}, 0, m) = M(\beta^{-1}\gamma^{-1}(\alpha\beta^{-1}\gamma^{-1})^{m-1})$ for all $m \in \mathbb{Z}^+$. By extending the values of μ this way, it follows that we can view the string modules belonging to 1-tubes as band modules. This allows us to treat all modules in 1-tubes in a uniform way.

Proposition 3.3. *Let $\Lambda = kQ/I$ be a basic algebra of dihedral type of polynomial growth, and let B be the unique band for Λ . Let $m \in \mathbb{Z}^+$, and let $\mu \in k \cup \{\infty\}$ (resp. $\mu \in k$, resp. $\mu \in k^*$) according to Q having 1 (resp. 2, resp. 3) vertices. Then the stable endomorphism ring of $M(B, \mu, m)$ is isomorphic to k if and only if $m = 1$ and $\Omega(M(B, \mu, 1)) \not\cong M(B, \mu, 1)$. More precisely:*

- (i) *If $\Lambda \in \{D(1)_0, D(2\mathcal{A})_0, D(3\mathcal{A})_1, D(3\mathcal{K})\}$ then $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\text{char}(k) \neq 2$ and $\mu \in k^*$.*
- (ii) *If $\Lambda \in \{D(1)_1, D(2\mathcal{A})_1\}$ then $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\mu \in k$.*
- (iii) *If $\Lambda \in \{D(3\mathcal{A})_2, D(3\mathcal{B})_{2,2}, D(3\mathcal{D})_2, D(3\mathcal{L})\}$ then $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\mu^2 \neq -1$.*
- (iv) *If $\Lambda \in \{D(3\mathcal{B})_{2,1}, D(3\mathcal{Q})\}$ then $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\mu^2 \neq 1$.*

In all cases, if $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ then $R(\Lambda, M(B, \mu, 1)) \cong k[[t]]$.

Proof. By [12], it follows in all cases that if M is a module in a 1-tube whose stable endomorphism ring is isomorphic to k , then M has to belong to the boundary of the 1-tube.

Suppose first that Q has a unique vertex, i.e. $\Lambda = D(1)_c$ for $c \in \{0, 1\}$, where $c = 1$ only occurs when $\text{char}(k) = 2$. Then $B = \beta\alpha^{-1}$ and $M(\beta) = M(B, \infty, 1)$ always satisfies $\Omega(M(\beta)) \cong M(\beta)$ and $\underline{\text{End}}_\Lambda(M(\beta)) \not\cong k$. If $c = 0$ and $\text{char}(k) = 2$ then $\Omega(M(B, \mu, 1)) \cong M(B, \mu, 1)$, and $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \not\cong k$ for all $\mu \in k$. If $c = 0$ and $\text{char}(k) \neq 2$ then $\Omega(M(B, \mu, 1)) \cong M(B, -\mu, 1)$, and $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\mu \in k^*$. If $c = 1$ then $\Omega(M(B, \mu, 1)) \cong M(B, 1 - \mu, 1)$, and $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\mu \in k$. In all cases when $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$, we have $\text{Ext}_\Lambda^1(M(B, \mu, 1), M(B, \mu, 1)) \cong k$, which means that $R(\Lambda, M(B, \mu, 1))$ is isomorphic to a quotient algebra of $k[[t]]$. Define L_μ to be the free $k[[t]]$ -module of rank 2 with an ordered basis $\{B_0, B_1\}$ and define a Λ -module structure on L_μ by letting (the image of) α (resp. β) act on $\{B_0, B_1\}$ by $X_\alpha = E_{01}$ (resp. $X_\beta = (\mu + t)E_{01}$). Here E_{01} denotes the 2×2 matrix which sends B_1 to B_0 and B_0 to 0. Then L_μ is a $k[[t]] \otimes_k \Lambda$ -module which is free as a $k[[t]]$ -module, $L_\mu/tL_\mu \cong M(B, \mu, 1)$ as Λ -modules, and L_μ/t^2L_μ defines a non-trivial lift of $M(B, \mu, 1)$ over $k[[t]]/(t^2)$. This implies that $R(\Lambda, M(B, \mu, 1)) \cong k[[t]]$.

Next suppose that Q has two vertices, i.e. $\Lambda = D(2\mathcal{A})_c$ for $c \in \{0, 1\}$, where $c = 1$ only occurs when $\text{char}(k) = 2$. Then $B = \alpha\beta^{-1}\gamma^{-1}$. If $c = 0$ and $\text{char}(k) = 2$ then $\Omega(M(B, \mu, 1)) \cong M(B, \mu, 1)$, and $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \not\cong k$ for all $\mu \in k$. If $c = 0$ and $\text{char}(k) \neq 2$ then $\Omega(M(B, \mu, 1)) \cong M(B, -\mu, 1)$, and $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\mu \in k^*$. If $c = 1$ then $\Omega(M(B, \mu, 1)) \cong M(B, 1 - \mu, 1)$, and $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\mu \in k$. In all cases when $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$, we have $\text{Ext}_\Lambda^1(M(B, \mu, 1), M(B, \mu, 1)) \cong k$, which means that $R(\Lambda, M(B, \mu, 1))$ is isomorphic to a quotient algebra of $k[[t]]$. Define L_μ to be the free $k[[t]]$ -module of rank 4 with an ordered basis $\{B_0, B_1, B_2, B_3\}$ and define a Λ -module structure on L_μ by letting (the image of) α (resp. β , resp. γ) act on $\{B_0, B_1, B_2, B_3\}$ by $X_\alpha = (\mu + t)E_{01}$ (resp. $X_\beta = E_{21}$, resp. $X_\gamma = E_{02}$). Here E_{ji} denotes the 3×3 matrix which sends B_i to B_j and all other basis elements to 0. Then L_μ is a $k[[t]] \otimes_k \Lambda$ -module which is free as a $k[[t]]$ -module, $L_\mu/tL_\mu \cong M(B, \mu, 1)$ as Λ -modules, and L_μ/t^2L_μ defines a non-trivial lift of $M(B, \mu, 1)$ over $k[[t]]/(t^2)$. This implies that $R(\Lambda, M(B, \mu, 1)) \cong k[[t]]$.

Next suppose that Q has precisely three vertices and that $\Lambda \in \{D(3\mathcal{A})_1, D(3\mathcal{K})\}$. If $\Lambda = D(3\mathcal{A})_1$ then $B = \beta\gamma\delta^{-1}\eta^{-1}$, and if $\Lambda = D(3\mathcal{K})$ then $B = \beta\kappa^{-1}\delta\gamma^{-1}\lambda\eta^{-1}$. If $\text{char}(k) = 2$ then $\Omega(M(B, \mu, 1)) \cong M(B, \mu, 1)$, and $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \not\cong k$ for all $\mu \in k^*$. If $\text{char}(k) \neq 2$ then $\Omega(M(B, \mu, 1)) \cong M(B, -\mu, 1)$, and $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\mu \in k^*$. In all cases when $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$, it follows similarly to the case $D(2\mathcal{A})_c$ that $R(\Lambda, M(B, \mu, 1)) \cong k[[t]]$.

Next suppose that Q has precisely three vertices and that $\Lambda \in \{D(3\mathcal{A})_2, D(3\mathcal{B})_{2,2}, D(3\mathcal{D})_2, D(3\mathcal{L})\}$. If $\Lambda = D(3\mathcal{A})_2$ then $B = \beta\gamma\delta^{-1}\eta^{-1}$, if $\Lambda = D(3\mathcal{B})_{2,2}$ then $B = \alpha\beta^{-1}\gamma^{-1}$, if $\Lambda = D(3\mathcal{D})_2$ then $B = \alpha\beta^{-1}\eta\xi^{-1}\delta\gamma^{-1}$, and if $\Lambda = D(3\mathcal{L})$ then $B = \alpha\beta^{-1}\delta^{-1}\lambda^{-1}$. We have $\Omega(M(B, \mu, 1)) \cong$

$M(B, -1/\mu, 1)$, and $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\mu^2 \neq -1$. If $\mu^2 \neq -1$ then it follows similarly to the case $D(2\mathcal{A})_c$ that $R(\Lambda, M(B, \mu, 1)) \cong k[[t]]$.

Finally suppose that Q has precisely three vertices and that $\Lambda \in \{D(3\mathcal{B})_{2,1}, D(3\mathcal{Q})\}$. If $\Lambda = D(3\mathcal{B})_{2,1}$ then $B = \alpha\beta^{-1}\eta\delta\gamma^{-1}$, if $\Lambda = D(3\mathcal{Q})$ then $B = \alpha\beta^{-1}\rho\delta^{-1}\lambda^{-1}$. We have $\Omega(M(B, \mu, 1)) \cong M(B, 1/\mu, 1)$, and $\underline{\text{End}}_\Lambda(M(B, \mu, 1)) \cong k$ if and only if $\mu^2 \neq 1$. If $\mu^2 \neq 1$ then it follows similarly to the case $D(2\mathcal{A})_c$ that $R(\Lambda, M(B, \mu, 1)) \cong k[[t]]$. \square

3.4. Blocks which are representation infinite of polynomial growth. Let G be a finite group and let B be a block of kG which is representation infinite of polynomial growth. By [11, Thm. 4.1], this occurs if and only if $\text{char}(k) = 2$ and the defect groups of B are Klein four groups. In this case, B is Morita equivalent to either $D(1)_0$, or $D(3\mathcal{A})_1$, or $D(3\mathcal{K})$. Using Propositions 3.1, 3.2 and 3.3, we obtain the following result.

Corollary 3.4. *Let $\Lambda = kQ/I$ be a basic algebra of dihedral type of polynomial growth. Then $R(\Lambda, V)$ is finite dimensional over k for all finitely generated Λ -modules V with $\underline{\text{End}}_\Lambda(V) \cong k$ if and only if $\text{char}(k) = 2$ and Λ is Morita equivalent to either $D(1)_0$, or $D(2\mathcal{A})_0$, or $D(3\mathcal{A})_1$, or $D(3\mathcal{K})$.*

In particular, suppose G is a finite group and B is a block of kG . Then B is representation infinite of polynomial growth if and only if B is Morita equivalent to an algebra Λ of dihedral type of polynomial growth and $R(\Lambda, V)$ is finite dimensional over k for all finitely generated Λ -modules V with $\underline{\text{End}}_\Lambda(V) \cong k$.

Note that $D(2\mathcal{A})_0$ is not Morita equivalent to a block of a group ring in characteristic 2. The reason is that if it were Morita equivalent to a block B , then the defect groups of B would have to have cardinality 4 and hence they would have to be Klein four groups. However, all blocks with Klein four defect groups are Morita equivalent to either $D(1)_0$, or $D(3\mathcal{A})_1$, or $D(3\mathcal{K})$.

4. APPENDIX: STRING AND BAND MODULES

Let k be an algebraically closed field, and let $\Lambda = kQ/I$ be a basic algebra of dihedral type of polynomial growth as in Figure 2. Then $\overline{\Lambda} = \Lambda/\text{soc}(\Lambda) = kQ/J$ is a string algebra. In particular, all non-projective indecomposable Λ -modules are inflated from string and band modules for $\overline{\Lambda}$. In this appendix, we give a brief introduction to these string and band modules. For more details, see [8].

For each arrow ζ in Q with starting vertex $s(\zeta)$ and end vertex $e(\zeta)$, we define a formal inverse ζ^{-1} with $s(\zeta^{-1}) = e(\zeta)$ and $e(\zeta^{-1}) = s(\zeta)$. A word w is a sequence $w_1 \cdots w_n$, where w_i is either an arrow or a formal inverse such that $s(w_i) = e(w_{i+1})$ for $1 \leq i \leq n-1$. Define $s(w) = s(w_n)$, $e(w) = e(w_1)$ and $w^{-1} = w_n^{-1} \cdots w_1^{-1}$. For each vertex u in Q , there is also an empty word 1_u of length 0 with $e(1_u) = u = s(1_u)$ and $(1_u)^{-1} = 1_u$. Denote the set of all words by \mathcal{W} .

Definition 4.1. Define an equivalence relation \sim on \mathcal{W} by $w \sim w'$ if and only if $w = w'$ or $w^{-1} = w'$. A string is a representative $w \in \mathcal{W}$ of an equivalence class under \sim_s with the following property: Either $w = 1_u$ for some vertex u in Q , or $w = w_1 \cdots w_n$ for some $n \geq 1$ where $w_i \neq w_{i+1}^{-1}$ for $1 \leq i \leq n-1$ and no subword of w or its formal inverse belongs to J .

If $C = 1_u$ for some vertex u in Q , then the string module $M(1_u)$ is defined to be the simple Λ -module S_u corresponding to u . Let now $C = w_1 \cdots w_n$ be a string of length $n \geq 1$, and define $v(i) = e(w_{i+1})$ for $0 \leq i \leq n-1$ and $v(n) = s(w_n)$. Then the string module $M(C)$ is defined to have an ordered k -basis $\{z_0, z_1, \dots, z_n\}$ such that the Λ -action on $M(C)$ is given by letting (the images of) each vertex u and each arrow ζ of Q act on $\{z_0, z_1, \dots, z_n\}$ as the following $(n+1) \times (n+1)$ matrices X_u and X_ζ , respectively: X_u sends z_i to itself if $v(i) = u$ and X_u sends z_i to 0 otherwise; whereas X_ζ sends z_i to z_{i-1} (resp. z_{i+1}) if $w_i = \zeta$ (resp. $w_{i+1} = \zeta^{-1}$) and X_ζ sends z_i to 0 otherwise.

Definition 4.2. For each $\Lambda = kQ/I$ as in Figure 2, there exists a unique band B as follows:

- If $\Lambda = D(1)_c$, then $B = \beta\alpha^{-1}$.

- If $\Lambda = D(2\mathcal{A})_c$, then $B = \alpha\beta^{-1}\gamma^{-1}$.
- If $\Lambda \in \{D(3\mathcal{A})_1, D(3\mathcal{A})_2\}$, then $B = \beta\gamma\delta^{-1}\eta^{-1}$.
- If $\Lambda = D(3\mathcal{B})_{2,1}$, then $B = \alpha\beta^{-1}\eta\delta\gamma^{-1}$.
- If $\Lambda = D(3\mathcal{B})_{2,2}$, then $B = \alpha\beta^{-1}\gamma^{-1}$.
- If $\Lambda = D(3\mathcal{D})_2$, then $B = \alpha\beta^{-1}\eta\xi^{-1}\delta\gamma^{-1}$.
- If $\Lambda = D(3\mathcal{K})$, then $B = \beta\kappa^{-1}\delta\gamma^{-1}\lambda\eta^{-1}$.
- If $\Lambda = D(3\mathcal{L})$, then $B = \alpha\beta^{-1}\delta^{-1}\lambda^{-1}$.
- If $\Lambda = D(3\mathcal{Q})$, then $B = \alpha\beta^{-1}\rho\delta^{-1}\lambda^{-1}$.

Let $B = w_1 \cdots w_n$ be the band for $\Lambda = kQ/I$ as above, and define $v(i) = e(w_{i+1})$ for $0 \leq i \leq n-1$. Then for each integer $m > 0$ and each $\mu \in k^*$, the band module $M(B, \mu, m)$ is defined to have an ordered k -basis

$$(4.1) \quad \{z_{0,1}, z_{0,2}, \dots, z_{0,m}, z_{1,1}, \dots, z_{1,m}, \dots, z_{n-1,1}, \dots, z_{n-1,m}\}$$

such that the Λ -action on $M(B, \mu, m)$ is given by letting (the images of) each vertex u and each arrow ζ of Q act on the basis in (4.1) as the following $nm \times nm$ matrices X_u and X_ζ , respectively: X_u sends $z_{i,j}$ to itself if $v(i) = u$ and X_u sends $z_{i,j}$ to 0 otherwise; whereas X_ζ sends $z_{i,j}$ to $\mu z_{0,j} + z_{0,j+1}$ (resp. $z_{i-1,j}$, resp. $z_{i+1,j}$) if $w_i = \zeta$ and $i = 1$ (resp. $w_i = \zeta$ and $i > 1$, resp. $w_{i+1} = \zeta^{-1}$) and X_ζ sends $z_{i,j}$ to 0 otherwise. Note that we set $z_{0,m+1} = 0 = z_{1,m+1}$ and $z_{n,j} = z_{0,j}$ for all j .

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